



## On $n$ -adic Algebra

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### Highlights

- **$n$ -adic algebra is a generalization of monadic algebra by extending the number of quantifiers from 1 to  $n$ .**
- **The method used in deduction of monadic algebra is generalized to the case of  $n$ -adic algebra with respect to terms.**

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### ABSTRACT

This paper studies  $n$ -adic algebras with terms. Deduction is also done in these algebras.

## 1. Introduction

$n$ -adic algebra is a generalization of monadic algebra by extending the number of quantifiers from 1 to  $n$ . Our main results are given in theorems 12, 13. The method used in (Alzubaidy, 2007) is generalized to the case of  $n$ -adic algebra. A Boolean algebra is an algebraic structure  $B = \langle A, \vee, \wedge, ', 0, 1 \rangle$  in which  $\vee, \wedge$  are binary operations on  $A$ ,  $'$  is a unary operation, while 0 and 1 are nullary operations (distinguished elements of  $A$ ), which satisfy the following conditions: For arbitrary  $a, b, c \in A$

- $a \vee b = b \vee a, a \wedge b = b \wedge a$
- $(a \vee b) \vee c = a \vee (b \vee c), (a \wedge b) \wedge c = a \wedge (b \wedge c)$
- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
- $a \vee (a \wedge b) = a, a \wedge (a \vee b) = a$
- $a \vee a' = 1, a \wedge a' = 0.$

A partial order  $\leq$  on a Boolean algebra is defined by  $a \leq b$ , if  $a \wedge b = a$  or  $a \vee b = b$ .

A Boolean algebra  $B$  is said to be complete if any subset  $A$  of  $B$  has infimum and supremum in  $B$ .

Suppose that  $B$  is a complete Boolean algebra. An existential operator on  $B$  is a mapping  $\exists: B \rightarrow B$  such that:

- $\exists(0) = 0$
- $a \leq \exists(a)$  for any  $a \in B$
- $\exists(a \wedge \exists(b)) = \exists(a) \wedge \exists(b)$  for any  $a, b \in B$

$(B, \exists)$  is called monadic algebra.

## 2. $n$ -adic algebras

An  $n$ -adic algebra  $(n \geq 0)(B, \exists_1, \dots, \exists_n)$  consists of a complete Boolean algebra  $B$  and  $n$  mappings  $\exists_i: B \rightarrow B; 1 \leq i \leq n$  called existential operators such that:

- $\exists_i(0) = 0$
- $a \leq \exists_i(a)$  for any  $a \in B$
- $\exists_i((a) \wedge \exists_i(b)) = \exists_i(a) \wedge \exists_i(b)$  for any  $a, b \in B$
- $\exists_i \exists_j = \exists_j \exists_i.$

Note that 0- $n$ -adic algebra is just Boolean algebra  $B$  and 1- $n$ -adic algebra is the monadic algebra  $(B, \exists)$ . Obviously, the  $n$ -adic algebra is a locally finite polyadic algebra in the sense given in (Alzubaidy and Bennour, 2012).

The universal operator  $\forall_i: B \rightarrow B; 1 \leq i \leq n$  is defined by  $\forall_i(a) = (\exists_i(a'))'$ , for any  $a \in B$ .

**Proposition 1.** (Alzubaidy, 2007)

- $\forall_i(0) = 0$
- $a \leq \forall_i(a)$  for any  $a \in B$
- $\forall_i((a) \wedge \forall_i(b)) = \forall_i(a) \wedge \forall_i(b)$  for any  $a, b \in B$
- $\forall_i \forall_j = \forall_j \forall_i.$

**Proposition 2.** (Alzubaidy, 2007)

$\exists_i \forall_j = \forall_j \exists_i$  if  $i \neq j$ .

Suppose  $Q_i$  is an existential operator or universal operator. By the previous proposition, we have

**Corollary 3.**

$Q_1 Q_2 \dots Q_n \leq \forall \forall \dots \exists \exists \dots \exists.$

## 2.1 Functional $n$ -adic Algebra

Let  $X$  be a nonempty set and  $B$  a Boolean algebra. Suppose that

$B^{X^n} = \{p | p: X^n \rightarrow B \text{ is a function}\}$ , where  $X^n$  is the Cartesian product of  $n$  copies of  $X$ .

For  $p, q \in B^{X^n}$  define  $p \wedge q, p \vee q, p'$  and  $0, 1$  pointwise as follows:

$$(p \wedge q)(x_1, \dots, x_n) = p(x_1, \dots, x_n) \wedge q(x_1, \dots, x_n),$$

$$(p \vee q)(x_1, \dots, x_n) = p(x_1, \dots, x_n) \vee q(x_1, \dots, x_n), p'(x_1, \dots, x_n) = (p(x_1, \dots, x_n))', 0(x_1, \dots, x_n) = 0 \text{ and } 1(x_1, \dots, x_n) = 1 \text{ for any } (x_1, \dots, x_n) \in X^n.$$

**Proposition 4.** (Alzubaidy, 2007)

$(B^{X^n}, \vee, \wedge, ', 0, 1)$  is an Boolean algebra.

**Proposition 5.** (Alzubaidy, 2007)

$(B^{X^n}, \exists)$  is a monadic algebra.

Existential and universal operators  $\exists_i, \forall_i$  on the functional Boolean algebra  $B^{X^n}$  are defined as follows:

$$\text{for each } p \in B^{X^n}, \exists_i(p)(x_1, \dots, x_n) = \sup_i \{p(x_1, \dots, x_n): (x_1, \dots, x_n) \in X^n\} \text{ and}$$

$$\forall_i(p)(x_1, \dots, x_n) = \inf_i \{p(x_1, \dots, x_n): (x_1, \dots, x_n) \in X^n\}.$$

**Proposition 6.**

$(B^{X^n}, \exists_i)$  is a  $n$ -adic algebra,  $1 \leq i \leq n$

**Proof**

$$\text{i) } \exists_i(0)(x_1, \dots, x_n) = \sup_i \{0(x_1, \dots, x_n): (x_1, \dots, x_n) \in X^n\} =$$

$$0(x_1, \dots, x_n) = 0. \text{ Then } \exists_i(0) = 0$$

$$\text{ii) Let } p \in B^{X^n}, \exists_i(p)(x_1, \dots, x_n) = \sup_i \{p(x_1, \dots, x_n): (x_1, \dots, x_n) \in X^n\} \geq p(x_1, \dots, x_n). \text{ Then } \exists_i(p) \geq p$$

$$\text{iii) Let } p, q \in B^{X^n}, \exists_i(p \wedge \exists_i(q))(x_1, \dots, x_n) = \sup_i \{p \wedge$$

$$\exists_i(q)(x_1, \dots, x_n): (x_1, \dots, x_n) \in X^n\} = \sup_i \left\{ (p)(x_1, \dots, x_n) \wedge \sup_i \{q(x_1, \dots, x_n): (x_1, \dots, x_n) \in X^n\} \right\} = \sup_i$$

$$\{p(x_1, \dots, x_n): (x_1, \dots, x_n) \in X^n\} \wedge \sup_i \{q(x_1, \dots, x_n): (x_1, \dots, x_n) \in X^n\} = \exists_i(p)(x_1, \dots, x_n) \wedge \exists_i(q)(x_1, \dots, x_n). \text{ Then } \exists_i(p \wedge \exists_i(q)) =$$

$$\exists_i(p) \wedge \exists_i(q)$$

$$\text{iv) } \exists_i(p)(x_1, \dots, x_n) \exists_j(p)(x_1, \dots, x_n) = \sup_i \{p(x_1, \dots, x_n)\} \sup_j$$

$$\{p(x_1, \dots, x_n)\} = \sup_j \{p(x_1, \dots, x_n)\} \sup_i \{p(x_1, \dots, x_n)\} =$$

$$\exists_j(p)(x_1, \dots, x_n) \exists_i(p)(x_1, \dots, x_n). \text{ Then } \exists_i \exists_j = \exists_j \exists_i.$$

A more general case is the locally finite polyadic algebra. This is given in (Alzubaidy, 2007).

## 3. Terms

Let  $X$  be a set of variables and  $\mathcal{F}$  be a type of algebra. The set  $T(X)$  of terms of type  $\mathcal{F}$  over  $X$  is the smallest set such that: i)  $X \cup \mathcal{F}_0 \subseteq T(X)$

ii) If  $t_1, \dots, t_n \in T(X)$  and  $f \in \mathcal{F}_n$  then  $f(t_1, \dots, t_n) \in T(X)$  (Burris and Shakappanavar, 1981).

For  $t \in T(X)$  we often write  $t$  as  $t(x_1, \dots, x_n)$  to indicate that the variables occurring in  $t$  are among  $x_1, \dots, x_n$ .

A term  $t$  is  $n$ -ary if the number of variables appearing explicitly is less than or equal  $n$ .

The term  $t$  define a function  $t_A: A^n \rightarrow A$  as  $t_A(a_1, \dots, a_n) = t(x_i/a_i)$  for  $a_1, \dots, a_n \in A$ , where  $A$  is an algebra of type  $\mathcal{F}$ .

The set  $T(X)$  can be transformed into an algebra (Burris and Shakappanavar, 1981). The term algebra  $T(X)$  of type  $\mathcal{F}$  over  $X$  has as its universe the set  $T(X)$  and the fundamental operations satisfy:

$$f^{T(X)}: (t_1, \dots, t_n) \mapsto f(t_1, \dots, t_n) \text{ for } f \in \mathcal{F}_n \text{ and } t_i \in T(X), 1 \leq i \leq n.$$

Now consider the functional  $n$ -adic algebra  $(B^{X^n}, \exists_i)$ . A term  $t$  defines a function  $t_*: B^{X^n} \rightarrow B^{X^n}$  as follows:

$$t_*(p)(x) = p(t_1(x), \dots, t_n(x)), \text{ for any } x \in X \text{ and } p \in B^{X^n} \text{ where } t = (t_1, \dots, t_n) \text{ as functions.}$$

$\text{End}(B^{X^n})$  is the set of an endomorphisms from  $B^{X^n}$  into itself.

**Proposition 7.**

$$\text{i) } t_* \in \text{End}(B^{X^n}) \quad \text{ii) } \exists_i t_* = t_* \exists_i$$

**Proof**

$$\text{i) } 1) t_*(p \vee q)(x) = (p \vee q)(t_1(x), \dots, t_n(x)) = p(t_1(x), \dots, t_n(x)) \vee q(t_1(x), \dots, t_n(x)) = t_*(p)(x) \vee t_*(q)(x).$$

$$\text{Then } t_*(p \vee q) = t_*(p) \vee t_*(q).$$

$$2) t_*(p \wedge q)(x) = (p \wedge q)(t_1(x), \dots, t_n(x)) = p(t_1(x), \dots, t_n(x)) \wedge q(t_1(x), \dots, t_n(x)) = t_*(p)(x) \wedge t_*(q)(x).$$

$$\text{Then } t_*(p \wedge q) = t_*(p) \wedge t_*(q).$$

$$3) (t_*(p)(x))' = (p(t_1(x), \dots, t_n(x)))' = p'(t_1(x), \dots, t_n(x)) = t_*(p')(x). \text{ Then } (t_*(p))' = t_*(p)'$$

$$4) t_*(0)(x) = 0(t_1(x), \dots, t_n(x)) = 0 = 0(x). \text{ Then } t_*(0) = 0.$$

$$5) t_*(1)(x) = 1(t_1(x), \dots, t_n(x)) = 1 = 1(x). \text{ Then } t_*(1) = 1.$$

$$\text{ii) } t_* \exists_i(p)(x) = t_* \left( \sup_i \{p(x)\} \right) = \sup_i \{p(t_1(x), \dots, t_n(x))\} = \sup_i \{t_*(p)(x)\} = \exists_i t_*(p)(x). \text{ Then } \exists_i t_* = t_* \exists_i.$$

## 4. Deduction

### 4.1 $n$ -adic ideals and filters

A subset  $U$  of a Boolean algebra  $B$  is called a Boolean ideal if

$$\text{i) } 0 \in U$$

$$\text{ii) } a \vee b \in U \text{ for any } a, b \in U$$

$$\text{iii) If } a \in U \text{ and } b \leq a \text{ then } b \in U.$$

A subset  $F$  of a Boolean algebra  $B$  is called a Boolean filter if

$$\text{i) } 1 \in F$$

$$\text{ii) } a \wedge b \in F \text{ for any } a, b \in F$$

$$\text{iii) If } a \in F \text{ and } b \geq a \text{ then } b \in F.$$

A subset  $U$  of an  $n$ -adic algebra  $B$  is called an  $n$ -adic ideal of  $B$  if

$$\text{i) } U \text{ is a Boolean ideal}$$

$$\text{ii) } \exists_i(U)(a) \in U, a \in U \text{ for } 1 \leq i \leq n.$$

A subset  $F$  of an  $n$ -adic algebra  $B$  is called an  $n$ -adic filter of  $B$  if

- i)  $F$  is a Boolean filter
- ii)  $\forall_i(F)(a) \in F, a \in F$  for  $1 \leq i \leq n$ .

Thus we have the following two propositions.

*Proposition 8.*

There is a one to one correspondence between ideals and filters.

*Proposition 9.*

The set of all  $n$ -adic ideals and the set of all  $n$ -adic filters are closed under the arbitrary intersection.

Let  $B$  be an  $n$ -adic algebra and  $\Gamma \subseteq B$ . Let  $U(\Gamma)$  denote the least  $n$ -adic ideal containing  $\Gamma$  and  $F(\Gamma)$  denote the least  $n$ -adic filter containing  $\Gamma$ . We say that  $U(\Gamma)$  and  $F(\Gamma)$  are generating by  $\Gamma$ .

*Proposition 10.*

Let  $B$  be an  $n$ -adic algebra and  $\Gamma \subseteq B$ . Then

- i)  $U(\Gamma) = \{b \in B: b \leq x_1 \vee x_2 \vee \dots \vee x_n \text{ for some } x_1, \dots, x_n \in \Gamma\} \cup \{0\}$
- ii)  $F(\Gamma) = \{b \in B: b \geq x_1 \wedge x_2 \wedge \dots \wedge x_n \text{ for some } x_1, \dots, x_n \in \Gamma\} \cup \{1\}$

*Proof*

- i) Let  $J = \{b \in B: b \leq x_1 \vee x_2 \vee \dots \vee x_n \text{ for some } x_1, \dots, x_n \in \Gamma\} \cup \{0\}$ ,  $0 \in J$ . Let  $b_1, b_2 \in J$  then  $b_1 \leq x_1 \vee x_2 \vee \dots \vee x_n$  and  $b_2 \leq y_1 \vee y_2 \vee \dots \vee y_m$  for some  $x_i, y_i \in \Gamma$ .

$b_1 \vee b_2 \leq x_1 \vee x_2 \vee \dots \vee x_n \vee y_1 \vee y_2 \vee \dots \vee y_m$ . Therefore  $b_1 \vee b_2 \in J$ .

If  $a \leq b \leq x_1 \vee x_2 \vee \dots \vee x_n$ , then  $a \in J$ . Then  $J$  is a Boolean ideal containing  $\Gamma$ .

If  $a \in J$ , then  $a \leq x_1 \vee x_2 \vee \dots \vee x_n$ ,  $\exists_i(J)(a) \leq \exists_i(J)(x_1 \vee x_2 \vee \dots \vee x_n) = \exists_i(J)(x_1) \vee \dots \vee \exists_i(J)(x_n)$ . Then  $\exists_i(J)(a) \in U(\Gamma)$ .

- ii) A similar argument leads to (ii).

A filter  $F$  of an  $n$ -adic algebra is called ultrafilter if  $F$  is maximal with respect to the property that  $0 \notin F$ .

Ultrafilters satisfy the properties of the following proposition.

*Proposition 11.* (Burris and Shakappanavar, 1981)

Let  $F$  be a filter of  $n$ -adic algebra, then

- i)  $F$  is an ultrafilter of  $B$  iff for any  $a \in F$  exactly one of  $a, a'$  belong to  $F$ .
- ii)  $F$  is an ultrafilter of  $B$  iff  $0 \in F$  and  $a \vee b \in F$  iff  $a \in F$  or  $b \in F$  for any  $a, b \in F$ .
- iii) If  $a \in B - F$ , then there is an ultrafilter  $L$  such that  $F \subseteq L$  and  $a \notin L$ .

Let  $\Gamma \subseteq B$ , the ultrafilter containing  $F(\Gamma)$  is denoted by  $UF(\Gamma)$ .

A mapping  $\mu: B_1 \rightarrow B_2$  between two  $n$ -adic algebras is called  $n$ -adic homomorphism if

- i)  $\mu$  is a Boolean homomorphism,
- ii)  $\mu \exists_i = \exists_i \mu$ .

Obviously,  $\mu \forall_i = \forall_i \mu$ .

For  $\Gamma \subseteq B, b \in B$ , we define the deduction  $\Gamma \vdash b$  iff  $b \in UF(\Gamma)$ .

*Theorem 12.*

Let  $\Gamma \subseteq B^{X^n}$

- i)  $\Gamma \vdash p \wedge q$  iff  $\Gamma \vdash p$  and  $\Gamma \vdash q$
- ii)  $\Gamma \vdash p \vee q$  iff  $\Gamma \vdash p$  or  $\Gamma \vdash q$
- iii)  $\Gamma \vdash p$  iff  $\Gamma \not\vdash p'$
- iv)  $\Gamma \vdash p$  iff  $\Gamma \vdash \forall_i p, 1 \leq i \leq n$
- v)  $F(\Gamma) \vdash p(x_0)$  iff  $F(\Gamma) \vdash \exists_i p(x), 1 \leq i \leq n$ .

*Proof*

(i), (ii) follow from the definition of filter and (Alzubaidy, 2007, p.2)

iii) follows from proposition 10.

iv) This is by  $\forall_i p(x) \leq p(x)$  and  $\forall_i(F) \subseteq F$ .

v) This is by the definition  $\exists_i p(x) = \sup\{p(x)\}, x = (x_1, \dots, x_n)$ .

Note that  $t_*(UF(\Gamma)) = UF(t_*(\Gamma))$ . Define  $\Gamma \vdash pt$  if  $pt \in F(t_*(\Gamma))$ . Then theorem 12 can be generalized with respect to terms as follows:

*Theorem 13.*

- i)  $\Gamma \vdash pt \wedge qt$  iff  $\Gamma \vdash pt$  and  $\Gamma \vdash qt$
- ii)  $\Gamma \vdash pt \vee qt$  iff  $\Gamma \vdash pt$  or  $\Gamma \vdash qt$
- iii)  $\Gamma \vdash pt$  iff  $\Gamma \not\vdash p't$
- iv)  $\Gamma \vdash p$  iff  $\Gamma \vdash \forall_i pt, 1 \leq i \leq n$
- v)  $F(\Gamma) \vdash pt(x_0)$  iff  $F(\Gamma) \vdash \exists_i pt(x), 1 \leq i \leq n$ .

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