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A comparative study on the behavior of Riemann-Liouville and Caputo fractional derivatives of some functions.

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Highlights

- **Fractional calculus is as old as Classical calculus.**
- **The geometrical behavior of the fractional derivatives of Riemann-Liouville and Caputo reveals both resembles and differences that may have some physical interpretations.**
- **Computational results of the fractional derivatives of various elementary functions are presented in graphical and tabular formats.**

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ABSTRACT

This paper presents an overview of fractional order derivative operators. Particular attention is devoted to the Riemann-Liouville and Caputo fractional derivative operators. A comparative study of these two frameworks to show how they behave geometrically. The computation results of some elementary function derivatives of fractional order are shown in graphic form and tabular for this purpose. The conclusion will include a few observations about derivatives of integer and fr **Abdlgader M. Inbaig, Yasmina M. Bashon**

actional order.

1. Introduction

Classical and fractional calculus are both ancient disciplines. The history begins in 1695, when Leibniz asked l'Hopital a crucial question about the order of the derivative in a letter: What is a possible order 1/2 derivative? (Oldham, 1974). Up until the middle of the previous century, a number of mathematicians have made significant contributions since then. Numerous researchers found the new theory to be very attractive, and it has been developed up to this point.

There are many different operators of the fractional derivatives that have been introduced and considered by Several authors and accepted in the literature, for additional information see (Miller, 1993); (Ishteva, 2005); and (Herzallah, Mohamed AE, 2014). For example, in their review of some of the most frequently used operators (Garrappa, 2021) and colleagues presented two approaches for extending integer-order derivatives to fractional order. They studied the effects of numerous derivatives on the same function and addressed specific properties of each fractional derivative in order to give a guide for the evaluation of fractional integrals and derivatives of some elementary functions.

A survey on the history of fractional calculus is presented by (Ishteva, 2005) in her Master's thesis; the main focus of her work was devoted to introducing the properties and applications of Caputo derivative operator. A comparison between the Caputo and Riemann-Liouville derivatives in terms of their properties is presented. general results are given by discussing a number of examples of fractional derivatives of some elementary functions. Historical notes and an introduction to fractional calculus are given by (Mainardi, 2013) as lectures to provide the essentials of fractional calculus according to different approaches that can be useful for some applications in the theory of probability and stochastic processes.

M AE Herzallah introduced the main properties of the Caputo, Riemann-Liouville, and Caputo via Riemann-Liouville fractional derivatives from a different perspective. He also provided some counterexamples to certain properties presented in some recent literatures and used to solve fractional nonlinear partial differential equations (Herzallah, Mohamed AE, 2014). Similarly, M. Labade offered an explanation of the definitions of the Riemann-Liouville and Caputo fractional derivatives, as well as a study of the differences between them (Labade, 2021).

In the same vein, we will compare the most well-known frameworks of fractional derivatives, namely, Riemann-Liouville and Caputo, in order to demonstrate their interrelation and geometrical behaviors in this study. However, we will restrict ourselves to a specified fractional order and domain. The MATLAB toolbox is used to display the computational results of the fractional derivatives of various simple functions in graphical formats and tables. In the conclusion, some general remarks on integer and fraction derivatives will be provided.

The paper is organized as follows: Section 2 presents the preliminaries, basic definitions and adopted notations. In addition, we will introduce the fractional derivative in Riemann-Liouville sense and Caputo sense, respectively. In Section 3, some of the most important properties of fractional derivatives and

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some general remarks. The core of the paper is found in Section 4, where we investigate applying the definitions of both Riemann-Liouville and Caputo fractional derivatives on some elementary functions separately. The graph of these functions and their fractional derivatives has been plotted using MATLAB and presented in this section as well. Some numerical results of applying both Riemann-Liouville and Caputo definitions on various functions are illustrated in an ad hoc Appendix. Finally, Section 5 outlines some brief remarks and conclusions.

2. Preliminaries and Basic Definitions

The gamma and beta functions, two fundamental analytical functions that are utilized in the majority of definitions of fractional order derivative operators, are briefly discussed in this section. We will present their definitions and address the basic properties of each one. In addition, the under-studied definitions of Riemann–Liouville and Caputo fractional derivatives, which are the most frequently used definitions for fractional derivatives, are presented here as well.

2.1 Gamma Function

The gamma function Γ *is* one of the functions that play an important role in the theory of differentiation and generalizes the ordinary definition of factorial of an integer number n and allows n to also take any non-negative integer (Davis, 1972). The integral transform definition for $\Gamma(x)$ is given by:

$$\Gamma(x) = \int_0^\infty y^{x-1}e^{-y}dy; \qquad x > 0 \tag{1}$$

and satisfies the following properties:

$$\Gamma(1) = 1, \Gamma(x+1) = x\Gamma(x), \text{and } \forall n \in N, \Gamma(n) = (n-1)!.$$

2.2 Beta Function

The beta function β is an important relation in fractional calculus. It is also, can be defined by means of integral as follows:

$$\beta(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}dt; Re(u) > 0, Re(u) > 0 \tag{2}$$

This integral is often called Beta integral. The most important property of Beta function is that it can be rewritten in terms of gamma function as follows (Ishteva, 2005):

$$\beta(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \quad Re(u) > 0, \quad Re(u) > 0. \tag{3}$$

2.3 Riemann-Liouville Fractional Derivative Operator

Let α be a real number, n be a positive integer, and $f(x)$. Then

$$D_L^\alpha(f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1}f(t)dt \tag{4}$$

The fractional derivative can be also alternatively defined in the Caputo sense by changing the order of the fractional integral and the nth–order derivative;

2.4 Caputo Fractional Derivative Operator

For $n-1 < \alpha < n$, Caputo proposed the following definition:

$$D_C^\alpha(f(x)) = \frac{1}{\Gamma(n-\alpha)}\int_0^\infty (x-t)^{n-\alpha-1} f^n(t)dt \tag{5}$$

3. Properties of Fractional Derivative Operators

In this section, we will discuss some of the most common properties of fractional derivative operators. The emphasis will be on the Riemann-Liouville and Caputo derivatives operators, as stated in the introduction.

3.1 Interpolation

Lemma 3.1 *Let $n-1 < \alpha < n, n \in \mathbb{N}, \alpha, \lambda \in \mathbb{R}$ and $f(t)$ be such that $D_C^\alpha f(t)$ exists. Then the following properties for the Caputo hold, where;*

$$\lim_{\alpha\to n^-} D_C^\alpha f(t) = f^{(n)} \tag{6}$$

$$\lim_{\alpha\to(n-1)^+} D_C^\alpha f(t) = f^{(n-1)}(t) - f^{(n-1)}(0) \tag{7}$$

The proof is achieved by using integration by parts and Beta function (Fakhouri, 2017).

For the Riemann-Liouville fractional differential operator, the interpolation of the corresponding properties is as follows:

$$\lim_{\alpha\to n^-} D_L^\alpha f(t) = f^{(n)}(t) \tag{8}$$

$$\lim_{\alpha\to(n-1)^+} D_L^\alpha f(t) = f^{(n-1)}(t) \tag{9}$$

where $n-1 < \alpha < n$.

3.2 Linearity

A direct characteristic of both Caputo and Riemann-Liouville operators is linearity;

Lemma 3.2 (Fakhouri, 2017) *Let $n-1 < \alpha < n, n \in \mathbb{N}, \alpha, \lambda \in \mathbb{R}$ and let $f(t)$ and $g(t)$ be functions such that D_C^α and D_L^α exist. Then Caputo fractional differential operator is linear, i.e.;*

$$\lim_{\alpha\to n} D_C^\alpha\left(\lambda f(t) + g(t)\right) = \lambda D_C^\alpha f(t) + D_C^\alpha g(t) \tag{10}$$

Similarly, the Riemann-Liouville operator satisfies

$$\lim_{\alpha\to n} D_L^\alpha\left(\lambda f(t)(t) + g(t)\right) = \lambda D_L^\alpha f(t) + D_L^\alpha g(t) \tag{11}$$

3.3 Non-Commutativity

Apply (Eq. 5) twice for any two orders, we observe;

Lemma 3.3 (Fakhouri, 2017) *Suppose that Suppose that $n-1 < \alpha < n$, $m, n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and the function $f(t)$ is such that $D_C^\alpha f(t)$ exists. Then we have the following:*

$$D_C^\alpha D_C^m f(t) = D_C^{\alpha+m} f(t) \neq D_C^m D_C^\alpha f(t) \tag{12}$$

Also, the Riemann-Liouville operator is not commutative and satisfies;

$$D_L^\alpha D_L^m f(t) = D_L^{\alpha+m} f(t) \neq D_L^m D_L^\alpha f(t) \tag{13}$$

It is noteworthy that two operators Riemann-Liouville and Caputo donot always coincide, unless if $f^{(s)}(0) = 0$; $s = 0, 1, \ldots, n-1$. So, we have the following:

Lemma 3.4: *The relation between Riemann-Liouville and Caputo fractional derivatives is given as follows:*

$$D_C^\alpha\left(f(x)\right) = D_L^\alpha\left(f(x) - f(0)\right) \tag{14}$$

As a result, then Riemann-Liouville and Caputo fractional derivatives coincide when $f(0) = 0$.

Example 3.5 *Let $\alpha = 0.2$, $m = 1$, and $f(t) = tsin(t)$, then*

$D_C^{0.2}D\ (tsin(t)) = D_C^{0.2+1}(tsin(t)) \neq DD_C^{0.2}\ (tsin(t))$, *where*

$$D_C^{0.2}\left(D\left(tsin(t)\right)\right)|_{t=\frac{\pi}{4}} = 1.358261$$

$$D_C^{1.2}\ tsin(t)|_{t=\frac{\pi}{4}} = 1.358261$$

$$D\left(D_C^{0.2}tsin(t)\right)|_{t=\frac{\pi}{4}} = 1.358261$$

However, when $m = 2$, we get

$$D_C^{0.2}\left(D^{(2)}tsin(t)\right)|_{t=\frac{\pi}{4}} = D_C^{2.2}\ tsin(t)|_{t=\frac{\pi}{4}} = -1.414204$$

$$D_C^{(2)}\left(D^{0.2}tsin(t)\right)|_{t=\frac{\pi}{4}} = -1.463095.$$

We remark that in the first case when $m = 1$, $f^{(s)}(0) = 0$; $s = 0, 1, \ldots, n-1$ (here $n = 1$), whereas, $m = 1$, give us $f^{(3)}(0) = 0$; $s = 0, 1, \ldots, n-1$; $(n = 2)$

Now, let us take another example;

Example 3.6

Let $\alpha = 0.2$, $m = 1$, and $f(t) = t + cos(t)$, then

$D_C^{0.2}D\left(t + cos(t)\right) = D_C^{0.2+1}\left(t + cos(t)\right) \neq DD_C^{0.2}\left(t + cos(t)\right)$, *where,*

$$D_C^{0.2}D\big(t+cos(t)\big)\big|_{t=\frac{\pi}{4}} = -0.780290$$
$$D_C^{1.2}\big(t+cos(t)\big)\big|_{t=\frac{\pi}{4}} = -0.780290$$
$$DD_C^{0.2}\big(t+cos(t)\big)\big|_{t=\frac{\pi}{4}} = 0.121163$$

4. Fractional derivatives for Some Special Functions

If we accept the perspective that the fractional derivative operator is a transformation of the function f into a new function g with new properties different from the original function, we can find a sensible geometric interpretation of the behavior of the fractional derivative of some functions.

In this section, we will compare the behavior of the Riemann-Liouville and Caputo fractional derivative formulas for a given function $f(x)$ in two cases (see Tables (1-12) in the Appendix):

Case 1: when α approaches zero; $\alpha = 0.1, \alpha = 0.01, \alpha = 0.001, \alpha = 0.0001, and\ \alpha = 0$

Case 2: when α approaches one; $\alpha = 0.9, \alpha = 0.99, \alpha = 0.999, \alpha = 0.9999, and\ \alpha = 1,$

The primary focus of our investigation will be the applying the derivatives on the following functions:

1. $f(x) = 1$
2. $f(x) = x$
3. $f(x) = x^2$
4. $f(x) = \sin(x)$
5. $f(x) = \cos(x)$
6. $f(x) = e^x$

The Riemann-Liouville and Caputo fractional derivatives of these functions have been plotted using MATLAB code, as seen in the following figures:

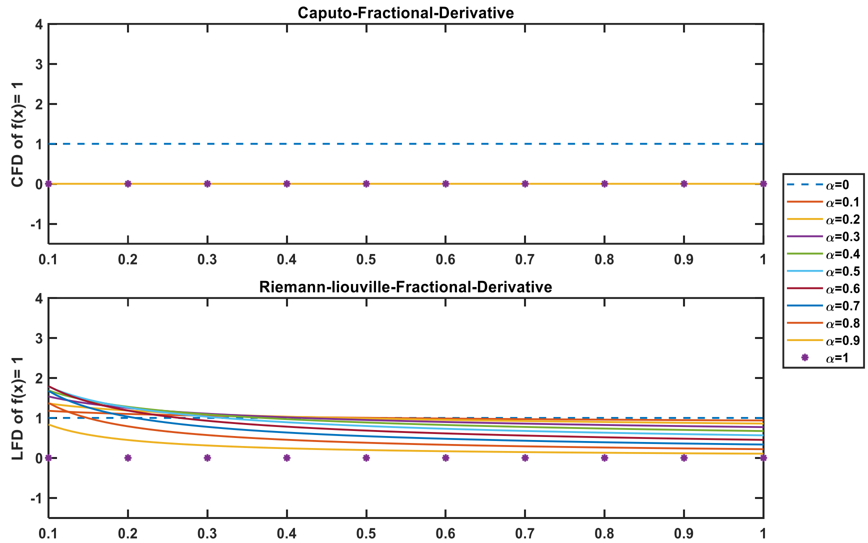
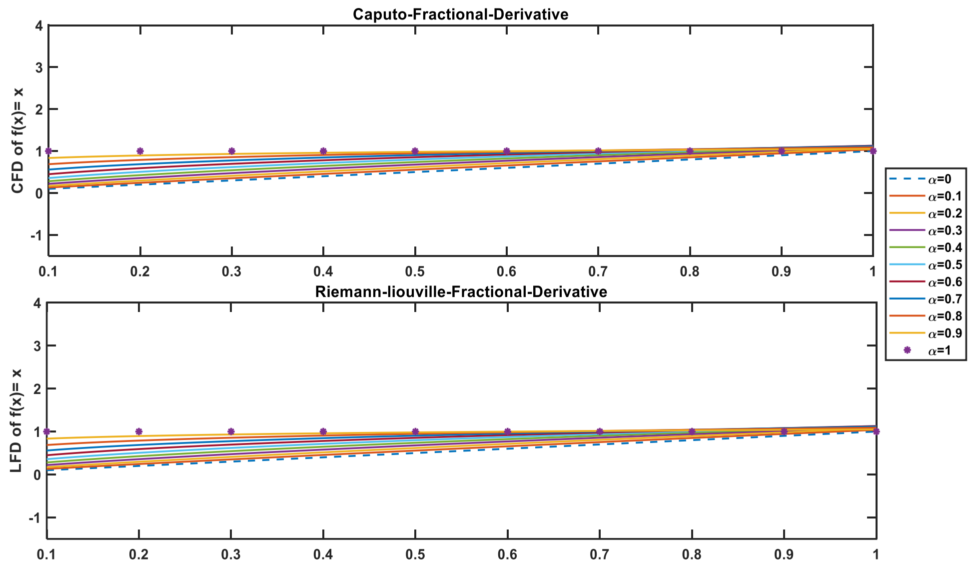


Fig. 1. Fractional derivative of $f(x) = 1$.



Fi g. 2. Fractional derivative of $f(x) = x$.

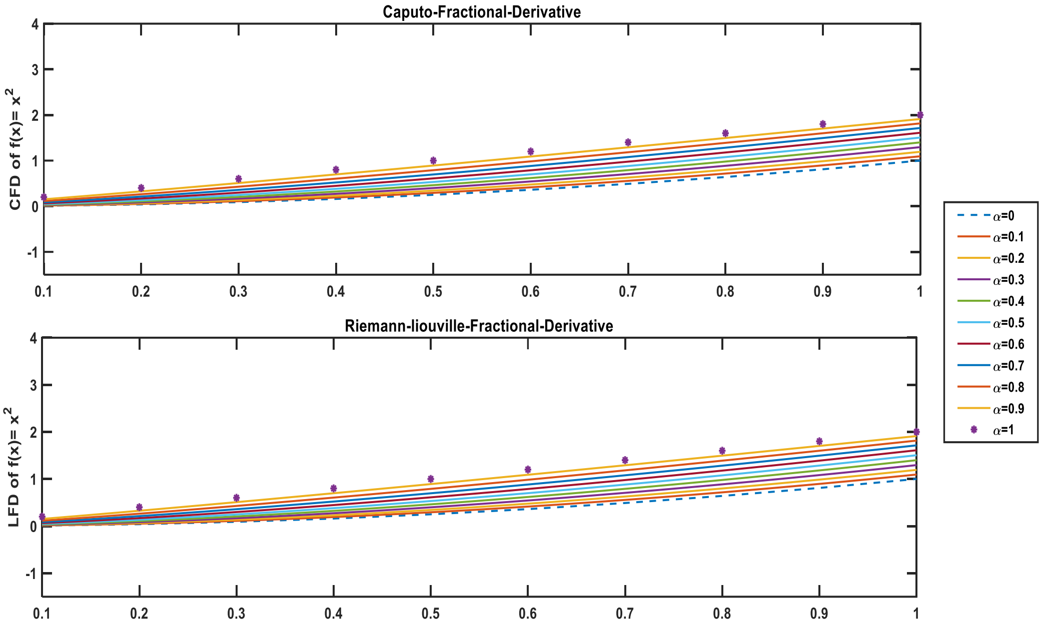


Fig. 3. Fractional derivative of $f(x) = x^2$

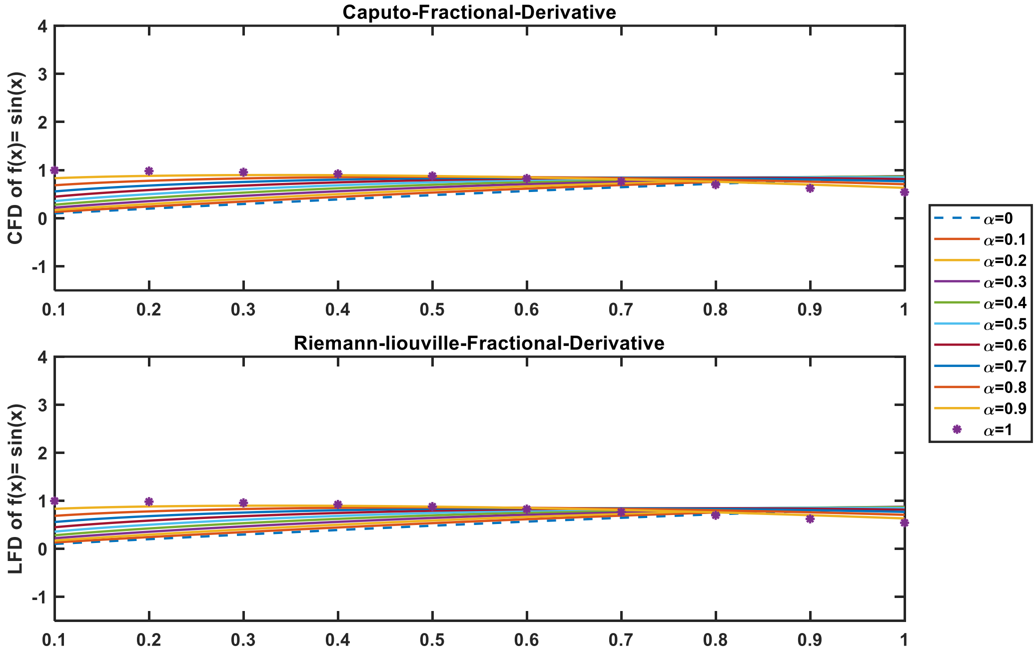


Fig. 4. Fractional derivative of $f(x) = \sin(x)$.

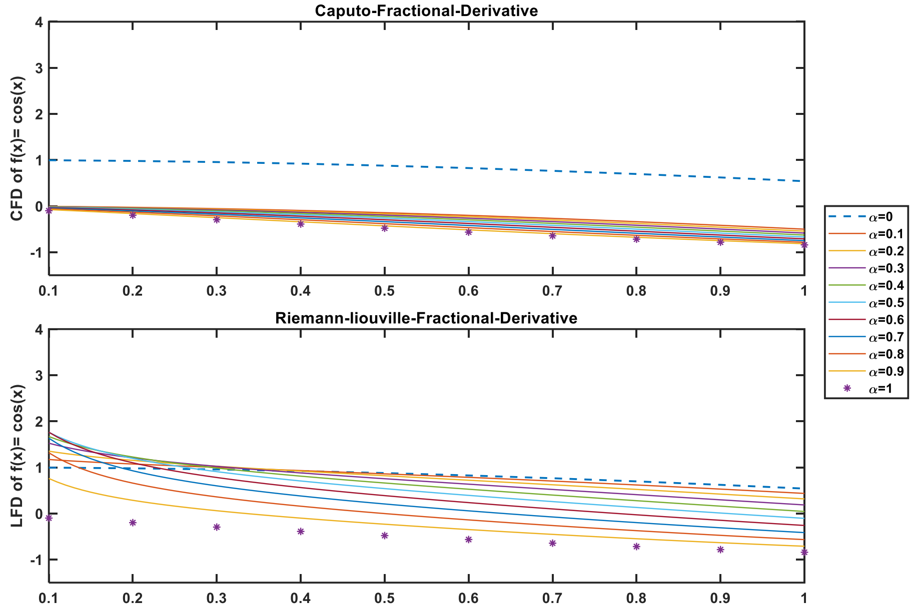


Fig. 5. Fractional derivative of $f(x) = \cos(x)$.

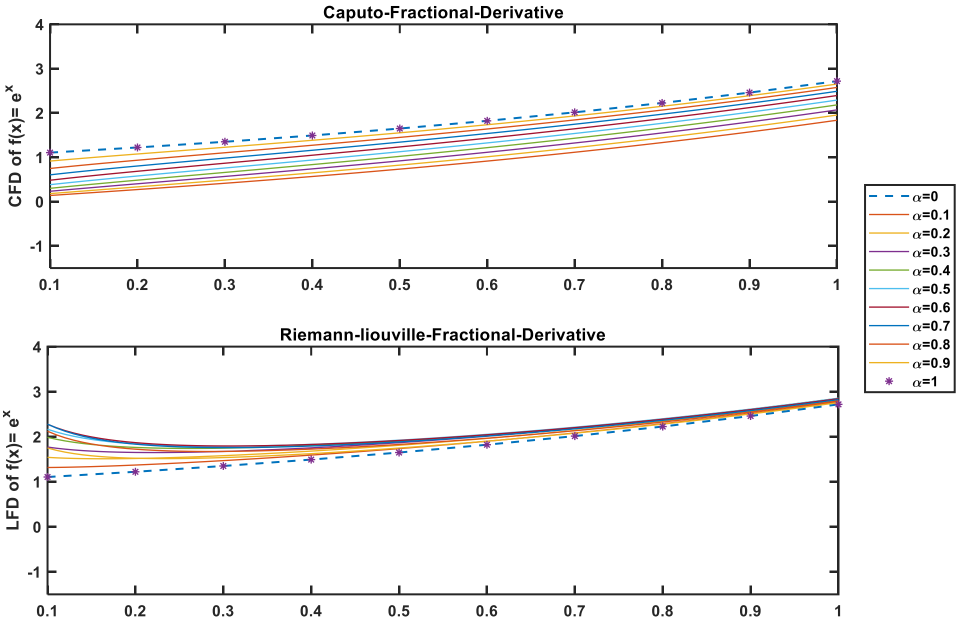


Fig. 6. Fractional derivative of $f(x) = \exp(x)$.

The numerical calculations presented in the Appendix demonstrate the differences between the Caputo and Riemann-Liouville derivatives of the elementary functions shown in the figures, when $0 \le x \le 1$ and $0 \le \alpha \le 1$. The Riemann-Liouville derivative has some drawbacks; for instant, The Riemann-Liouville fractional derivative of a constant k is not zero; $D_L^{\alpha} k \neq 0$: this can be shown in (Fig. 1.; Table. 1 and Table. 2). Moreover, the fractional derivation of the function $f(x) = \cos(x)$ has a singularity at $x = 0$, as well as for $f(x) = \exp(x)$, since $f(0) \neq 0$ (see Figs. 5. & 6.; Tables 9.,10.,11., 12.). This disadvantage restricts the usage of Riemann-Liouville fractional derivative.

In the other hand, for Caputo fractional derivative, in order to calculate function's fractional derivative, we must compute its derivative first: functions without a first-order derivative may have fractional derivatives of all orders less than one in the Riemann-Liouville sense, whereas Caputo derivatives are only defined for differentiable functions (we refer the reader to the definitions in Section 2; Eq. (4) and Eq. (5)).

It is noteworthy that both Riemann-Liouville and Caputo Fractional derivatives coincide when $f(0) = 0$; as shown in (Figs. 2, 3, 4), as well as (Tables 3 through 8, which illustrate the functions x, x^2, and $\sin(x)$ (see Lemma 3.4.; and Example 3.5).

5. Conclusion

The material we presented in this paper is not novel; there are numerous studies on the same topic as in (Miller, 1993); (Ishteva, 2005); and (Mainardi, 2013) and others. However, the focus of our work in this paper is on comparing the Riemann-Liouville and Caputo fractional derivatives and attempting to determine the geometric interpretation of each under certain conditions. We tried to explore their basic properties and provide some examples of some special functions for explaining the differences between them. The Riemann-Liouville fractional derivative has certain disadvantages when trying to apply the definition. These disadvantages reduce the field of application of the Riemann-Liouville fractional derivative. Caputo derivative demands higher conditions of regularity for differentially computing the fractional derivative of a function in the Caputo sense. Caputo derivatives are defined only for differentiable functions, while functions that have no first-order derivative might have fractional derivatives of all other orders less than one in Riemann-Liouville sense. On both graphical and numerical calculations, the procedure of calculating specific fractional derivatives of some fundamental functions and analyzing the impact of various order fractional derivatives on the same function produced intriguing results.

Appendix: Tables of Both Caputo and Riemann-Liouville fractional derivatives of some Special Functions when $0 \le x \le 1$ *and* $0 \le \alpha \le 1$.

Table 1

Caputo and Riemann-Liouville fractional derivatives of $f(x) = 1$ and $\alpha \to 0$.

			$f(x) = 1$			
α / x	FD	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.001$	$\alpha = 0.0001$	$\alpha = 0$
$x = 0$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 Inf	0.000000000 Inf	0.000000000 Inf	0.000000000 Inf	1.000000000 1.000000000
$x = 0.1$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 1.178075611	0.000000000 1.017319313	0.000000000 1.001726034	0.000000000 1.000172543	1.000000000 1.000000000
$x = 0.2$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 1.099183412	0.000000000 1.010292176	0.000000000 1.001031931	0.000000000 1.000103219	1.000000000 1.000000000
$x = 0.3$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 1.055506810	0.000000000 1.00620408	0.000000000 1.000626130	0.000000000 1.000062669	1.000000000 1.000000000
$x = 0.4$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 1.025574387	0.000000000 1.003313578	0.000000000 1.000338309	0.000000000 1.000033899	1.000000000 1.000000000
$x = 0..5$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 1.002942799	0.000000000 1.001077244	0.000000000 1.000115115	0.000000000 1.000011584	1.000000000 1.000000000
$x = 0.6$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 0.984822677	0.000000000 0.999253728	0.000000000 1.000115115	0.000000000 0.999993352	1.000000000 1.000000000
$x = 0.7$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 0.969757978	0.000000000 0.997714558	0.000000000 0.999778660	0.000000000 0.999977937	1.000000000 1.000000000
$x = 0.8$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 0.956894738	0.000000000 0.996383185	0.000000000 0.999645167	0.000000000 0.999964585	1.000000000 1.000000000
$x = 0.9$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 0.945690256	0.000000000 0.995210305	0.000000000 0.999527433	0.000000000 0.999952807	1.000000000 1.000000000
$x = 1.0$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 0.935778720	0.000000000 0.994162299	0.000000000 0.999422128	0.000000000 0.999942271	1.000000000 1.000000000

Table 2

Caputo and Riemann-Liouville fractional derivatives of $\boldsymbol{f(x) = 1}$ and $\boldsymbol{\alpha \to 1}$.

			$f(x) = 1$			
α / x	FD	$\alpha = 0.9$	$\alpha = 0.99$	$\alpha = 0.999$	$\alpha = 0.9999$	$\alpha = 1$
$x = 0$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 Inf	0.000000000 Inf	0.000000000 Inf	0.000000000 Inf	0.000000000 0.000000000
$x = 0.1$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 0.834947802	0.000000000 0.098281385	0.000000000 0.009982752	0.000000000 0.000999827	0.000000000 0.000000000
$x = 0.2$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 0.447437448	0.000000000 0.049482493	0.000000000 0.004994837	0.000000000 0.000499948	0.000000000 0.000000000
$x = 0.3$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 0.310634863	0.000000000 0.033122356	0.000000000 0.003331242	0.000000000 0.000333312	0.000000000 0.000000000
$x = 0.4$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 0.239775791	0.000000000 0.024913335	0.000000000 0.002499150	0.000000000 0.000249991	0.000000000 0.000000000
$x = 0..5$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 0.196149101	0.000000000 0.019975192	0.000000000 0.001999766	0.000000000 0.000199997	0.000000000 0.000000000
$x = 0.6$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 0.166465101	0.000000000 0.016676370	0.000000000 0.001666775	0.000000000 0.000166667	0.000000000 0.000000000
$x = 0.7$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 0.144900902	0.000000000 0.014316083	0.000000000 0.001428885	0.000000000 0.000142860	0.000000000 0.000000000
$x = 0.8$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 0.128492665	0.000000000 0.012543310	0.000000000 0.001250441	0.000000000 0.000125004	0.000000000 0.000000000
$x = 0.9$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 0.115568923	0.000000000 0.011162749	0.000000000 0.001111634	0.000000000 0.000111116	0.000000000 0.000000000
$x = 1.0$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 0.105113700	0.000000000 0.010057065	0.000000000 0.001000576	0.000000000 0.000100005	0.000000000 0.000000000

Table 3

Caputo and Riemann-Liouville fractional derivatives of $f(x) = x$ and $\alpha \to 0$.

$f(x) = x$						
α / x	FD	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.001$	$\alpha = 0.0001$	$\alpha = 0$
$x = 0$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	0.000000000 0.000000000	0.000000000 0.000000000	0.000000000 0.000000000	0.000000000 0.000000000	0.000000000 0.000000000
$x = 0.1$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	0.130897290 0.130897290	0.102759526 0.102759526	0.100272876 0.100272876	0.100027257 0.100027257	0.100000000 0.100000000
$x = 0.2$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	0.244262980 0.244262980	0.204099429 0.204099429	0.200406793 0.200406793	0.200040647 0.200040647	0.200000000 0.200000000
$x = 0.3$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	0.351835603 0.351835603	0.304910329 0.304910329	0.300488327 0.300488327	0.300048805 0.300048805	0.300000000 0.300000000
$x = 0.4$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	0.455810838 0.455810838	0.405379223 0.405379223	0.400535859 0.400535859	0.40005356 0.40005356	0.400000000 0.400000000
$x = 0..5$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	0.557190444 0.557190444	0.505594568 0.505594568	0.500558115 0.500558115	0.500055798 0.500055798	0.500000000 0.500000000
$x = 0.6$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	0.656548451 0.656548451	0.605608320 0.605608320	0.600560233 0.600560233	0.600056017 0.600056017	0.600000000 0.600000000
$x = 0.7$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	0.754256205 0.754256205	0.705454738 0.705454738	0.700545608 0.700545608	0.700054562 0.700054562	0.700000000 0.700000000
$x = 0.8$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	0.850573100 0.850573100	0.805158129 0.805158129	0.800516651 0.800516651	0.800051673 0.800051673	0.800000000 0.800000000
$x = 0.9$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	0.945690256 0.945690256	0.904736641 0.904736641	0.900475165 0.900475165	0.900047531 0.900047531	0.900000000 0.900000000
$x = 1.0$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	1.039754134 1.039754134	1.004204342 1.004204342	1.000422551 1.000422551	1.00004227 1.00004227	1.000000000 1.000000000

Table 4

Caputo and Riemann-Liouville fractional derivatives of $f(x) = x$ and $\alpha \to 1$.

$f(x) = x$						
α / x	FD	$\alpha = 0.9$	$\alpha = 0.99$	$\alpha = 0.999$	$\alpha = 0.9999$	$\alpha = 1$
$x = 0$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	0.000000000 0.000000000	0.000000000 0.000000000	0.000000000 0.000000000	0.000000000 0.000000000	1.000000000 1.000000000
$x = 0.1$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	0.834947802 0.834947802	0.982813853 0.982813853	0.998275297 0.998275297	0.999827469 0.999827469	1.000000000 1.000000000
$x = 0.2$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	0.894874897 0.894874897	0.989649863 0.989649863	0.998967489 0.998967489	0.999896774 0.999896774	1.000000000 1.000000000
$x = 0.3$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	0.93190459 0.93190459	0.993670694 0.993670694	0.999372617 0.999372617	0.999937318 0.999937318	1.000000000 1.000000000
$x = 0.4$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	0.959103167 0.959103167	0.996533423 0.996533423	0.999660160 0.999660160	0.999966084 0.999966084	1.000000000 1.000000000
$x = 0..5$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	0.980745505 0.9807455053	0.998759606 0.998759606	0.999883253 0.999883253	0.999988398 0.999988398	1.000000000 1.000000000
$x = 0.6$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	0.998790610 0.998790610	1.000582221 1.000582221	1.000065570 1.000065570	1.000006630 1.000006630	1.000000000 1.000000000
$x = 0.7$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	1.014306316 1.014306316	1.002125814 1.002125814	1.000219742 1.000219742	1.000022046 1.000022046	1.000000000 1.000000000
$x = 0.8$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	1.02794132 1.02794132	1.003464861 1.003464861	1.000353312 1.000353312	1.000035399 1.000035399	1.000000000 1.000000000
$x = 0.9$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	1.040120310 1.040120310	1.004647468 1.004647468	1.000471144 1.000471144	1.000047178 1.000047178	1.000000000 1.000000000
$x = 1.0$	$D_C^{\alpha} f(x)$ $D_L^{\alpha} f(x)$	1.051137006 1.051137006	1.005706528 1.005706528	1.000576559 1.000576559	1.000057715 1.000057715	1.000000000 1.000000000

Table 5

Caputo and Riemann-Liouville fractional derivatives of $f(x) = x^2$ and $\alpha \to 0$.

$f(x) = x^2$						
x \ α	FD	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.001$	$\alpha = 0.0001$	$\alpha = 0$
$x = 0$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 0.000000000	0.000000000 0.000000000	0.000000000 0.000000000	0.000000000 0.000000000	0.000000000 0.000000000
$x = 0.1$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.013778662 0.013778662	0.010327590 0.010327590	0.010032303 0.010032303	0.010003225 0.010003225	0.010000000 0.010000000
$x = 0.2$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.051423785 0.051423785	0.041025010 0.041025010	0.040101409 0.040101409	0.040010130 0.040010130	0.040000000 0.040000000
$x = 0.3$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.111105980 0.111105980	0.091932762 0.091932762	0.090191594 0.090191594	0.090019142 0.090019142	0.090000000 0.090000000
$x = 0.4$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.191920353 0.191920353	0.162966522 0.162966522	0.160294491 0.160294491	0.160029427 0.160029427	0.160000000 0.160000000
$x = 0..5$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.293258128 0.293258128	0.254067622 0.254067622	0.250404260 0.250404260	0.250040401 0.250040401	0.250000000 0.250000000
$x = 0.6$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.414662179 0.414662179	0.365190946 0.365190946	0.360516398 0.360516398	0.360051612 0.360051612	0.360000000 0.360000000
$x = 0.7$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.555767730 0.555767730	0.496299815 0.496299815	0.490627239 0.490627239	0.490062696 0.490062696	0.490000000 0.490000000
$x = 0.8$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.716272085 0.716272085	0.647363320 0.647363320	0.640733687 0.640733687	0.640073342 0.640073342	0.640000000 0.640000000
$x = 0.9$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.895917084 0.895917084	0.818354751 0.818354751	0.810833065 0.810833065	0.810083282 0.810083282	0.810000000 0.810000000
$x = 1.0$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	1.094478036 1.094478036	1.009250595 1.009250595	1.000923012 1.000923012	1.000092280 1.000092280	1.000000000 1.000000000

Table 6

Caputo and Riemann-Liouville fractional derivatives of $f(x) = x^2$ and $\alpha \to 1$.

$f(x) = x^2$						
x \ α	FD	$\alpha = 0.9$	$\alpha = 0.99$	$\alpha = 0.999$	$\alpha = 0.9999$	$\alpha = 1$
$x = 0$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 0.000000000	0.000000000 0.000000000	0.000000000 0.000000000	0.000000000 0.000000000	0.000000000 0.000000000
$x = 0.1$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.151808691 0.151808691	0.194616604 0.194616604	0.199455603 0.199455603	0.199945499 0.199945499	0.200000000 0.200000000
$x = 0.2$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.325409053 0.325409053	0.391940540 0.391940540	0.399187807 0.399187807	0.399918718 0.399918718	0.400000000 0.400000000
$x = 0.3$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.508311595 0.508311595	0.590299422 0.590299422	0.599024546 0.599024546	0.599902400 0.599902400	0.600000000 0.600000000
$x = 0.4$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.697529576 0.697529576	0.789333404 0.789333404	0.798929199 0.798929199	0.799892878 0.799892878	0.800000000 0.800000000
$x = 0..5$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.891586823 0.891586823	0.988870897 0.988870897	0.998884368 0.998884368	0.999888409 0.999888409	1.000000000 1.000000000
$x = 0.6$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	1.089589757 1.089589757	1.188810559 1.188810559	1.198879804 1.198879804	1.199887968 1.199887968	1.200000000 1.200000000
$x = 0.7$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	1.290935311 1.290935311	1.389085287 1.389085287	1.398908731 1.398908731	1.399890875 1.399890875	1.400000000 1.400000000
$x = 0.8$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	1.495187378 1.495187378	1.589647304 1.589647304	1.598966333 1.598966333	1.599896649 1.599896649	1.600000000 1.600000000
$x = 0.9$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	1.702015053 1.702015053	1.790460835 1.790460835	1.799049010 1.799049010	1.799904930 1.799904930	1.800000000 1.800000000
$x = 1.0$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	1.911158192 1.911158192	1.991498076 1.991498076	1.999153965 1.999153965	1.999915438 1.999915438	2.000000000 2.000000000

Table 7

Caputo and Riemann-Liouville fractional derivatives of $\boldsymbol{f(x) = sin(x)}$ and $\boldsymbol{\alpha \to 0}$.

$f(x) = sin(x)$						
x \ α	FD	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.001$	$\alpha = 0.0001$	$\alpha = 0$
$x = 0$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 0.000000000	0.000000000 0.000000000	0.000000000 0.000000000	0.000000000 0.000000000	0.000000000 0.000000000
$x = 0.1$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.130659851 0.130659851	0.102586911 0.102586911	0.100105699 0.100105699	0.09986061 0.09986061	0.099833416 0.099833416
$x = 0.2$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.242493454 0.242493454	0.202730109 0.202730109	0.199072307 0.199072307	0.198709597 0.198709597	0.198669330 0.198669330
$x = 0.3$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.346115748 0.346115748	0.300319121 0.300319121	0.295997510 0.295997510	0.295567911 0.295567911	0.295520206 0.295520206
$x = 0.4$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.442685336 0.442685336	0.394565719 0.394565719	0.389931229 0.389931229	0.389469612 0.389469612	0.389418342 0.389418342
$x = 0..5$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.532238275 0.532238275	0.484616607 0.484616607	0.479943629 0.479943629	0.479477337 0.479477337	0.479425538 0.479425538
$x = 0.6$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.614453343 0.614453343	0.569623940 0.569623940	0.565140470 0.565140470	0.564692271 0.564692271	0.564642473 0.564642473
$x = 0.7$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.688880114 0.688880114	0.648772585 0.648772585	0.644673885 0.644673885	0.644263310 0.644263310	0.644217687 0.644217687
$x = 0.8$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.7550341205 0.7550341205	0.721296000 0.721296000	0.717751611 0.717751611	0.717395658 0.717395658	0.717356090 0.717356090
$x = 0.9$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.8124453323 0.8124453323	0.786487689 0.786487689	0.783645283 0.783645283	0.783358769 0.783358769	0.783326909 0.783326909
$x = 1.0$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.8606864579 0.8606864579	0.843710335 0.843710335	0.841697910 0.841697910	0.841493707 0.841493707	0.841470984 0.841470984

Table 8

Caputo and Riemann-Liouville fractional derivatives of $\boldsymbol{f(x) = sin(x)}$ and $\boldsymbol{\alpha \to 1}$.

$f(x) = sin(x)$						
x \ α	FD	$\alpha = 0.9$	$\alpha = 0.99$	$\alpha = 0.999$	$\alpha = 0.9999$	$\alpha = 1$
$x = 0$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.166593989 0.166593989	0.836510255 0.836510255	0.982313979 0.982313979	0.998217236 0.998217236	1.000000000 1.000000000
$x = 0.1$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.831336153 0.831336153	0.977976654 0.977977147	0.993295549 0.993295549	0.994833236 0.994833236	0.995004165 0.995004165
$x = 0.2$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.879427932 0.879427932	0.970214855 0.970214855	0.979084444 0.979084444	0.979968376 0.979968376	0.980066578 0.980066578
$x = 0.3$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.895852976 0.895852976	0.949945966 0.949945966	0.954803808 0.954803808	0.955283256 0.955283256	0.955336489 0.955336489
$x = 0.4$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.893503762 0.893503762	0.91902836 0.91902836	0.920865596 0.920865596	0.921041495 0.921041495	0.921060994 0.921060994
$x = 0..5$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.876675264 0.876675264	0.878292102 0.878292102	0.877662002 0.877662002	0.877590543 0.877590543	0.877582562 0.877582562
$x = 0.6$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.847493047 0.847493047	0.828377005 0.828377005	0.825648357 0.825648357	0.825366919 0.825366919	0.825335615 0.825335615
$x = 0.7$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.80731565 0.80731565	0.769906253 0.769906253	0.765356986 0.765356986	0.764893687 0.764893687	0.764842187 0.764842187
$x = 0.8$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.75719268 0.75719268	0.703538489 0.703538489	0.697397835 0.697397835	0.69677583 0.69677583	0.696706709 0.696706709
$x = 0.9$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.698049282 0.698049282	0.629984936 0.629984936	0.622454791 0.622454791	0.621694445 0.621694445	0.621609968 0.621609968
$x = 1.0$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.630769878 0.630769878	0.550013447 0.550013447	0.541279978 0.541279978	0.540400055 0.540400055	0.540302306 0.540302306

Table 9

Caputo and Riemann-Liouville fractional derivatives of $\boldsymbol{f(x) = cos(x)}$ and $\boldsymbol{\alpha \to 0}$.

$f(x) = cos(x)$						
α / x	FD	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.001$	$\alpha = 0.0001$	$\alpha = 0$
$x = 0$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 Inf	0.000000000 Inf	0.000000000 Inf	0.000000000 Inf	1.000000000 1.00000000
$x = 0.1$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.00688324 1.171192357	-0.00515946 1.012159846	-0.00501197 0.996714064	-0.00499744 0.995175098	0.995004165 0.995004165
$x = 0.2$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.02562108 1.073562321	-0.02044382 0.989848354	-0.01998391 0.981048012	-0.01993846 0.980164753	0.980066578 0.980066578
$x = 0.3$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.05511230 1.000394506	-0.04562066 0.960583431	-0.04475840 0.955867735	-0.04467299 0.955389678	0.955336489 0.955336489
$x = 0.4$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.09410145 0.930964239	-0.08039629 0.922917293	-0.07908368 0.921254629	-0.07895346 0.921080437	0.921060994 0.921060994
$x = 0..5$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.14341583 0.859526966	-0.12439393 0.876683312	-0.12261389 0.877501225	-0.12243707 0.877574514	0.877582562 0.877582562
$x = 0.6$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.20081334 0.784009339	-0.17715145 0.822102278	-0.17491183 0.825020953	-0.17468911 0.825304235	0.540204458 0.540204458
$x = 0.7$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.26604693 0.703711052	-0.23812338 0.759591182	-0.23545314 0.764325519	-0.23518733 0.764790605	0.764842187 0.764842187
$x = 0.8$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.338313583 0.618581154	-0.30668506 0.689698126	-0.30363135 0.696013816	-0.30332709 0.696637500	0.696706709 0.696706709
$x = 0.9$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.416742275 0.528927981	-0.38213811 0.613072197	-0.37876392 0.620763518	-0.37842741 0.621525396	0.621609968 0.621609968
$x = 1.0$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.500496913 0.435281807	-0.46371688 0.530445422	-0.46009895 0.539323175	-0.45973781 0.540204458	0.540302306 0.540302306

Table 10

Caputo and Riemann-Liouville fractional derivatives of $\boldsymbol{f(x) = cos(x)}$ and $\boldsymbol{\alpha \to 1}$.

$f(x) = cos(x)$						
α / x	FD	$\alpha = 0.9$	$\alpha = 0.99$	$\alpha = 0.999$	$\alpha = 0.9999$	$\alpha = 1$
$x = 0$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	0.000000000 Inf	0.000000000 inf	0.000000000 Inf	0.000000000 inf	0.000000000 0.000000000
$x = 0.1$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.07578781 0.759159998	-0.09714754 0.00113384	-0.09956181 -0.08957905	-0.09980623 -0.09880640	-0.09983341 -0.09983341
$x = 0.2$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.16170671 0.285730732	-0.19467719 -0.14519470	-0.19826704 -0.19327220	-0.19862907 -0.19812912	-0.19866933 -0.19866933
$x = 0.3$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.25065722 0.059977649	-0.29077875 -0.25765639	-0.29504348 -0.29171223	-0.29547251 -0.29513920	-0.29552021 -0.29552021
$x = 0.4$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.340258344 -0.100482552	-0.38431222 -0.35939888	-0.38890587 -0.386406717	-0.38936708 -0.38911709	-0.38941834 -0.38941834
$x = 0..5$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.42887735 -0.23272825	-0.47425731 -0.45428211	-0.47890767 -0.47690790	-0.47937375 -0.47917374	-0.47942553 -0.47942553
$x = 0.6$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.51518237 -0.34871727	-0.55966469 -0.54298832	-0.56414451 -0.56247773	-0.56459268 -0.56442601	-0.56464247 -0.56464247
$x = 0.7$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.59800977 -0.45310887	-0.63964749 -0.62533140	-0.64376133 -0.64233245	-0.64417206 -0.64402920	-0.64421769 -0.64421769
$x = 0.8$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.676314104 -0.547821439	-0.71338265 -0.70083935	-0.71696023 -0.71570979	-0.71731652 -0.71719152	-0.71735609 -0.71735609
$x = 0.9$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.749147665 -0.633578742	-0.78101560 -0.76895285	-0.78300803 -0.78189639	-0.78329505 -0.78318393	-0.78332691 -0.78332691
$x = 1.0$	$D_C^\alpha f(x)$ $D_L^\alpha f(x)$	-0.815652758 -0.710539058	-0.83916564 -0.82910858	-0.84124340 -0.84024282	-0.84144826 -0.84134825	-0.84147099 -0.84147099

Table 11

Caputo and Riemann-Liouville fractional derivatives of $\boldsymbol{f(x) = exp(x)}$ and $\boldsymbol{\alpha \to 0}$.

$f(x) = exp(x)$						
x \ α	FD	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.001$	$\alpha = 0.0001$	$\alpha = 0$
$x = 0$	$D_C^\alpha f(x)$	0.000000000	0.000000000	0.000000000	0.000000000	1.000000000
	$D_L^\alpha f(x)$	1.775543112	1.064727695	1.006343818	1.000633097	1.000000000
$x = 0.1$	$D_C^\alpha f(x)$	0.138030402	0.108100441	0.105460557	0.1051998490	1.105170918
	$D_L^\alpha f(x)$	1.317291208	1.125522055	1.107196664	1.105373398	1.105170918
$x = 0.2$	$D_C^\alpha f(x)$	0.271842883	0.226055636	0.221864299	0.221448875	1.221402758
	$D_L^\alpha f(x)$	1.37157768	1.23639848	1.222901251	1.221552596	1.221402758
$x = 0.3$	$D_C^\alpha f(x)$	0.413606024	0.355857429	0.350454994	0.349918389	1.349858808
	$D_L^\alpha f(x)$	1.469465494	1.36209513	1.351084466	1.349981393	1.349858808
$x = 0.4$	$D_C^\alpha f(x)$	0.566483219	0.498949866	0.492533827	0.491895577	1.491824698
	$D_L^\alpha f(x)$	1.592314481	1.502288572	1.492874641	1.491929727	1.491824698
$x = 0..5$	$D_C^\alpha f(x)$	0.732702443	0.656824237	0.649528582	0.648801972	1.648721271
	$D_L^\alpha f(x)$	1.735846153	1.657921534	1.6496457	1.648813757	1.648721271
$x = 0.6$	$D_C^\alpha f(x)$	0.914273023	0.831090023	0.823013404	0.822208236	1.82211880
	$D_L^\alpha f(x)$	1.89926007	1.830360427	1.822947863	1.822201755	1.82211880
$x = 0.7$	$D_C^\alpha f(x)$	1.11320129	1.023507266	1.014726151	1.013850032	2.013752707
	$D_L^\alpha f(x)$	2.083097984	2.021236095	2.014506242	2.013828113	2.013752707
$x = 0.8$	$D_C^\alpha f(x)$	1.331586322	1.236010603	1.226586412	1.225645462	2.225540928
	$D_L^\alpha f(x)$	2.288600815	2.232406257	2.226232831	2.225610172	2.225540928
$x = 0.9$	$D_C^\alpha f(x)$	1.571675879	1.470731636	1.460715022	1.459714293	2.459603111
	$D_L^\alpha f(x)$	2.517471329	2.465953011	2.460243567	2.459667211	2.459603111
$x = 1.0$	$D_C^\alpha f(x)$	1.835907001	1.730021845	1.719455437	1.718399185	2.718281828
	$D_L^\alpha f(x)$	2.771779398	2.724194095	2.718878566	2.718341557	2.718281828

Table 12

Caputo and Riemann-Liouville fractional derivatives of $\boldsymbol{f(x) = exp(x)}$ and $\boldsymbol{\alpha \to 1}$.

$f(x) = exp(x)$						
x \ α	FD	$\alpha = 0.9$	$\alpha = 0.99$	$\alpha = 0.999$	$\alpha = 0.9999$	$\alpha = 1.0$
$x = 0$	$D_C^\alpha f(x)$	0.000000000	0.000000000	0.000000000	0.000000000	1.000000000
	$D_L^\alpha f(x)$	0.000000000	0.000000000	0.000000000	0.000000000	1.000000000
$x = 0.1$	$D_C^\alpha f(x)$	0.914586138	1.085128294	1.103157307	1.104969464	1.105170918
	$D_L^\alpha f(x)$	1.757124362	1.184392937	1.113240845	1.105979395	1.105170918
$x = 0.2$	$D_C^\alpha f(x)$	1.074125558	1.206482602	1.219909535	1.221253425	1.221402758
	$D_L^\alpha f(x)$	1.523587099	1.256211382	1.224929459	1.221755886	1.221402758
$x = 0.3$	$D_C^\alpha f(x)$	1.226155033	1.337612353	1.348636054	1.349736552	1.349858808
	$D_L^\alpha f(x)$	1.537725232	1.370844432	1.351978432	1.350070979	1.349858808
$x = 0.4$	$D_C^\alpha f(x)$	1.383777554	1.481308172	1.490776481	1.491719911	1.491824698
	$D_L^\alpha f(x)$	1.624094396	1.506283352	1.493281892	1.491970529	1.491824698
$x = 0..5$	$D_C^\alpha f(x)$	1.552110237	1.639444318	1.647797877	1.648628975	1.648721271
	$D_L^\alpha f(x)$	1.748613255	1.65945916	1.649801649	1.648829373	1.648721271
$x = 0.6$	$D_C^\alpha f(x)$	1.734351246	1.813785926	1.821290328	1.822036002	1.82211880
	$D_L^\alpha f(x)$	1.901066566	1.830489872	1.822959885	1.822202948	1.82211880
$x = 0.7$	$D_C^\alpha f(x)$	1.933091433	2.006168933	2.012999456	2.013677434	2.013752707
	$D_L^\alpha f(x)$	2.078178983	2.020505302	2.014430385	2.013820499	2.013752707
$x = 0.8$	$D_C^\alpha f(x)$	2.150750855	2.218569401	2.22484909	2.225471798	2.225540928
	$D_L^\alpha f(x)$	2.279388319	2.231128261	2.226101096	2.225596959	2.225540928
$x = 0.9$	$D_C^\alpha f(x)$	2.389767365	2.453143349	2.458962557	2.459539110	2.459603111
	$D_L^\alpha f(x)$	2.505452038	2.464318398	2.460075428	2.45965035	2.459603111
$x = 1.0$	$D_C^\alpha f(x)$	2.652696908	2.712257478	2.717684871	2.718222188	2.718281828
	$D_L^\alpha f(x)$	2.757905349	2.722324515	2.718686449	2.718322294	2.718281828

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